

KÄHLER-RICCI SOLITONS ON HOMOGENEOUS TORIC BUNDLES (II)

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ABSTRACT. It is proved that an homogeneous toric bundles over a flag manifold $G^{\mathbb{C}}/P$ admits a Kähler-Ricci solitonic metric if and only if it is Fano. In particular, an homogeneous toric bundle of this kind is Kähler-Einstein if and only if it is Fano and its Futaki invariant vanishes identically.

1. INTRODUCTION

In this paper we continue the discussion of [14] on Kähler-Einstein and Kähler-Ricci solitonic metrics over homogeneous bundles $\pi : M \rightarrow V$, with fiber equal to a compact toric Kähler manifold F and basis V equal to a generalized flag manifold $V = G^{\mathbb{C}}/P$ of a complex semisimple Lie group $G^{\mathbb{C}}$. We call any such bundle a *homogeneous toric bundle*.

In [14] we gave necessary and sufficient conditions in order that a homogeneous toric bundle $\pi : M \rightarrow V = G^{\mathbb{C}}/P$ has positive first Chern class; In particular this occurs only if F is Fano. In this second part we determine when a homogeneous toric bundle admits a Kähler-Ricci soliton.

We recall that a *Kähler-Ricci soliton* consists of a Kähler form ω associated with a (real) vector field X such that

$$\rho - \omega = \mathcal{L}_X \omega, \quad \mathcal{L}_{JX} \omega = 0,$$

where ρ denotes the Ricci form of ω . Notice that if the associated vector field X is trivial, the Kähler-Ricci soliton ω is a Kähler-Einstein form.

Our main result is the following.

Theorem 1.1. *Let F be a toric Kähler manifold of dimension m and $\pi : M \rightarrow V$ be a homogeneous toric bundle with fiber F and basis $V = G^{\mathbb{C}}/P$. The bundle M admits a Kähler-Ricci soliton if and only if it is Fano.*

In particular, the bundle M is Kähler-Einstein if and only if it is Fano and its Futaki functional vanishes identically.

This theorem extends the result of X.-J. Wang and X. Zhu ([19]) who proved the existence of a Kähler-Ricci soliton on any Fano toric manifold F , i.e. when the basis of the toric bundle reduces to a single point. On the other hand, our theorem includes the results of N. Koiso and Y. Sakane in [15, 9, 10], which give necessary and sufficient conditions for homogeneous toric bundles with fiber $\mathbb{C}P^1$ in order to

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be Kähler-Einstein (see also [13, 5]). It also generalizes Koiso's result ([8]) on the existence of a Kähler-Ricci soliton on any Fano, homogeneous toric bundle with fiber $\mathbb{C}P^1$ (see also [17]).

The paper is organized as follows. In §2 we fix notations and recall some facts on homogeneous toric bundles that were used and/or proved in [14]. In §3 we recall the fundamental results of [16, 17] on Kähler-Ricci solitons and we obtain some consequences on homogeneous toric bundles. In §4 we compute the holomorphic invariant introduced by Tian and Zhu in [17] and in §5 we show that the problem of finding a Kähler-Ricci soliton on the homogeneous toric bundles can be reduced to a suitable partial differential equation on the toric fiber F : This equation turns out to be very close to the equation studied in [19]. We conclude showing that under suitable modifications, the arguments used in the proof of Wang and Zhu for the solvability of that equation works in our case as well.

We remark that from the proof of Theorem 1.1, it follows that a vector field X on a homogeneous toric bundle over a flag manifold is the associated vector field of a Kähler-Ricci soliton if and only if the Tian and Zhu's invariant $\mathcal{F}_X(\cdot)$ vanishes identically.

2. NOTATIONS AND PRELIMINARIES

As we mentioned in the Introduction, this paper is the continuation of [14] and we will constantly use the same notation and definitions introduced in that paper. For readers convenience, we briefly recall here all notations and definitions adopted in that paper, but we refer to [14] for more detailed information.

For any Lie group G , we will denote its Lie algebra by the corresponding gothic letter \mathfrak{g} . Given a Lie homomorphism $\tau : G \rightarrow G'$, we will always use the same letter to represent the induced Lie algebra homomorphism $\tau : \mathfrak{g} \rightarrow \mathfrak{g}'$. The center of G will be denoted by $Z(G)$ and the center of \mathfrak{g} by $\mathfrak{z}(\mathfrak{g})$.

If G acts on a manifold N , for any $X \in \mathfrak{g}$, we will use the symbol \hat{X} to indicate the corresponding induced vector field on N . We recall here that $[\widehat{X}, \widehat{Y}] = -[\hat{X}, \hat{Y}]$ for every $X, Y \in \mathfrak{g}$.

We will also denote by N_{reg} the set of G -principal points in N .

The Cartan Killing form of a semisimple Lie algebra \mathfrak{g} will be always denoted by \mathcal{B} and, for any $X \in \mathfrak{g}$, we set $X^\vee = -\mathcal{B}(X, \cdot) \in \mathfrak{g}^*$. Given a root system R w.r.t. a fixed maximal torus, we will denote by $E_\alpha \in \mathfrak{g}^\mathbb{C}$ the root vector corresponding to the root α in the Chevalley normalization and by $H_\alpha = [E_\alpha, E_{-\alpha}]$ the \mathcal{B} -dual of α .

In all the following, F denotes a compact, toric Kähler manifold with $\dim_\mathbb{C} F = m$ and we indicate by T^m the m -dimensional torus acting effectively on F by holomorphic isometries. A homogeneous toric bundle is a compact Kähler manifold of the form

$$M = G^\mathbb{C} \times_{P, \tau} F = G \times_{K, \tau} F \quad (2.1)$$

where $V = G^\mathbb{C}/P = G/K$ is a flag manifold of (complex) dimension n , G is a compact semisimple Lie group, $G^\mathbb{C}$ its complexification, P a suitable parabolic subgroup and $\tau : P \rightarrow (T^m)^\mathbb{C}$ is a surjective homomorphism.

We will constantly identify F with the fiber $F = F_{eK} = \pi^{-1}(eK)$ over the base point $eK \in V = G/K$.

The complex structures of M , F and V will be denoted by J , J_F and J_V , respectively. Notice that J_V is the natural $G^{\mathbb{C}}$ -invariant complex structure of the complex homogeneous space $G^{\mathbb{C}}/P$ and that J is the unique $G^{\mathbb{C}}$ -invariant complex structure on M , which makes $\pi : M \rightarrow V$ a holomorphic map and induces on $F = \pi^{-1}(eK)$ the complex structures J_F .

We observe that both $G^{\mathbb{C}}$ and $(T^m)^{\mathbb{C}}$ act naturally as groups of holomorphic transformations on (M, J) , with two commuting actions. The action of $G^{\mathbb{C}}$ is the one induced on M by its standard action on $G^{\mathbb{C}} \times F$, while the action of $(T^m)^{\mathbb{C}}$ is defined by

$$h([g, x]_{K, \tau}) \stackrel{\text{def}}{=} [g, h^{-1}(x)]_{K, \tau}, \quad \text{for any } h \in (T^m)^{\mathbb{C}}.$$

For this reason, in the following we will identify $G^{\mathbb{C}} \times (T^m)^{\mathbb{C}}$ with the corresponding subgroup of $\text{Aut}(M, J)$ and $\mathfrak{g}^{\mathbb{C}} + \mathfrak{t}^{\mathbb{C}}$ will be identified with the corresponding subalgebra of $\mathfrak{aut}(M, J) = \text{Lie}(\text{Aut}(M, J))$.

We recall that \mathfrak{g} admits an $\text{Ad}(K)$ -invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and that, for any fixed CSA $\mathfrak{h} \subset \mathfrak{k}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, the associated root system R admits a corresponding decomposition $R = R_o + R_m$, so that $E_{\alpha} \in \mathfrak{k}^{\mathbb{C}}$ if $\alpha \in R_o$ and $E_{\alpha} \in \mathfrak{m}^{\mathbb{C}}$ if $\alpha \in R_m$. Furthermore, J_V induces a splitting $R_m = R_m^+ \cup R_m^-$ into two disjoint subset of positive and negative roots, so that the J_V -holomorphic and J_V -antiholomorphic subspaces of $\mathfrak{m}^{\mathbb{C}}$ are given by

$$\mathfrak{m}^{(1,0)} = \sum_{\alpha \in R_m^+} \mathbb{C}E_{\alpha}, \quad \mathfrak{m}^{(0,1)} = \sum_{\alpha \in R_m^-} \mathbb{C}E_{\alpha}. \quad (2.2)$$

The Lie algebra \mathfrak{p} of the parabolic subgroup P is $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{(0,1)}$.

We also recall that for any G -invariant Kähler form ω of V there exists a uniquely associated element $Z_{\omega} \in \mathfrak{z}(\mathfrak{k})$ so that $\omega(\hat{X}, \hat{Y})|_{eK} = \mathcal{B}(Z_{\omega}, [X, Y])$ for any $X, Y \in \mathfrak{g}$. In particular, the G -invariant Kähler-Einstein form ω_V on V , with Einstein constant $c = 1$, is associated with the element

$$Z_V = -\frac{1}{2\pi} \sum_{\alpha \in R_m^+} iH_{\alpha}. \quad (2.3)$$

(see e.g. [2, 3] - be aware that in this paper, we adopt the definition of Ricci form ρ used e.g. in [7], which differs from the one in [2] and [3] by the factor $\frac{1}{2\pi}$).

The homomorphism $\tau : P \rightarrow (T^m)^{\mathbb{C}}$ is completely determined by its restriction to the connected component of the identity $Z^o(K)$ of $Z(K)$, which gives a surjective homomorphism $\tau : Z^o(K) \rightarrow T^m$ and a surjective Lie algebra homomorphism $\tau : \mathfrak{z}(\mathfrak{k}) \rightarrow \mathfrak{t}$.

In the following, we will denote by $\mathfrak{t}_G \stackrel{\text{def}}{=} (\ker \tau)^{\perp} \cap \mathfrak{z}(\mathfrak{k})$. Notice that \mathfrak{t}_G integrates to a closed subtorus and moreover we can choose a \mathcal{B} -orthonormal basis (Z_1, \dots, Z_m) of \mathfrak{t}_G so that $\exp(\mathbb{R} \cdot Z_j)$ is closed for every $j = 1, \dots, m$. We will denote by $Z_j' \stackrel{\text{def}}{=} \tau(Z_j)$ for $j = 1, \dots, m$ and by ν_j the smallest real number such that $\exp(\nu_j Z_j') = e$.

We will also denote by $(F_{\alpha_1}, G_{\alpha_1}, \dots, F_{\alpha_n}, G_{\alpha_n})$ the basis for $\mathfrak{m} \subset \mathfrak{g}$ given by the element

$$F_{\alpha_i} = \frac{1}{\sqrt{2}}(E_{\alpha_i} - E_{-\alpha_i}), \quad G_{\alpha_i} = \frac{i}{\sqrt{2}}(E_{\alpha_i} + E_{-\alpha_i}), \quad \alpha_i \in R_{\mathfrak{m}}. \quad (2.4)$$

The rest of this section will be devoted to the properties we will later use of the so-called “algebraic representative” of a closed G -invariant 2-form and of their relations with the moment maps.

If ψ is a G -invariant closed 2-form on M , then there exists a G -equivariant map $Z_\psi : M \rightarrow \mathfrak{g}$, uniquely associated with ψ so that

$$\psi_p(\hat{X}, \hat{Y}) = \mathcal{B}([Z_\psi|_p, X], Y) = \mathcal{B}(Z_\psi|_p, [X, Y]) \quad \text{for any } X, Y \in \mathfrak{g} \quad (2.5)$$

This map is called *algebraic representative of ψ* and, in case ψ is non-degenerate, the moment map determined by ψ

$$\mu_\psi : M \rightarrow \mathfrak{g}^*$$

coincides with the $(-\mathcal{B})$ -dual map of Z_ψ

$$Z_\psi^\vee \stackrel{\text{def}}{=} -\mathcal{B}(Z_\psi, \cdot) : M \rightarrow \mathfrak{g}^*.$$

By G -equivariance, any algebraic representative Z_ψ is uniquely determined by its restriction on the fiber $F = \pi^{-1}(eK)$ and such restriction $Z_\psi|_F$ takes values in $\mathfrak{z}(\mathfrak{k})$.

In case a G -equivariant 2-form ψ is cohomologous to 0, its restriction to F must be of the form $\psi = dd^c\phi$ for some K -invariant smooth function $\phi : F \rightarrow \mathbb{R}$ and the restriction to F of its algebraic representative is

$$Z_\psi|_F = -\sum_i J\hat{Z}_i(\phi)Z_i. \quad (2.6)$$

For any given Kähler form $\omega \in c_1(M)$, the restrictions to F of the algebraic representatives of ω and of its Ricci form ρ are as follows:

$$Z_\omega|_F = \sum_i f_i Z_i + Z_V, \quad \text{for some smooth functions } f_i : F \rightarrow \mathbb{R}, \quad (2.7)$$

$$Z_\rho|_F = \sum_{i=1}^m \frac{J\hat{Z}_i(\log h)}{4\pi} Z_i + Z_V, \quad (2.8)$$

where $Z_V \in \mathfrak{z}(\mathfrak{k})$ is the element defined in (2.3) and

$$h = \det(-f_{i,j}) \cdot \prod_{\alpha \in R_{\mathfrak{m}}^+} (a_\alpha^i f_i + b_\alpha), \quad (2.9)$$

where $f_{i,j} \stackrel{\text{def}}{=} J\hat{Z}_j(f_i)$, $a_\alpha^i \stackrel{\text{def}}{=} \alpha(iZ_i)$, $b_\alpha \stackrel{\text{def}}{=} \alpha(iZ_V)$.

Moreover, for any $p \in F$,

$$-f_{i,j}(p) = \omega_p(\hat{Z}_j, J\hat{Z}_i) = \frac{1}{2\pi} g_p(\hat{Z}_i, \hat{Z}_j). \quad (2.10)$$

In other words, for any point $p \in F$, the values $-f_{i,j}(p)$ are the entries of a symmetric, positive definite matrix and one can check that the map

$$\mu : F \rightarrow \mathfrak{t}^*, \quad \mu(q) = -\mathcal{B}\left(\sum_\ell f_\ell Z_\ell, \cdot\right) \Big|_{\mathfrak{t}_G} \in \mathfrak{t}_G^* \simeq \mathfrak{t} \quad (2.11)$$

is a moment map for the action of T^m determined by $\omega|_{TF}$.

If F has $c_1(F) > 0$, for a given T^m -invariant Kähler form $\psi \in c_1(F)$, a corresponding moment map $\mu_\psi : F \rightarrow \mathfrak{t}^*$ is called *metrically normalized* if $\int_F \psi \cdot \eta_\psi^m = 0$ where η_ψ is the unique Kähler form in $c_1(F)$ that has ψ as Ricci form. By [14] §4, for any $\psi \in c_1(F)$, there exists a unique associated metrically normalized moment map and the polytope $\Delta_F = \mu_\psi(F)$ is independent of ψ and it is called the *canonical polytope of F* .

If M is Fano, by Thm. 1.1 of [14], then also F is Fano and the moment map defined in (2.11) is the metrically normalized moment map determined by $\omega|_{TF}$. In particular, $\mu(F) = \Delta_F$.

In all the following, we will also constantly identify $\mathfrak{t}^*(= \mathfrak{t}_G^*)$ with \mathbb{R}^m , through the vector space isomorphism that maps the elements $e_\ell \stackrel{\text{def}}{=} \frac{1}{4\pi} \mathcal{B}(Z_\ell, \cdot)|_{\mathfrak{t}_G}$ into the canonical basis of \mathbb{R}^m . By virtue of such identification, in next sections the map μ will always be written as

$$\mu : F \rightarrow \Delta_F \subset \mathbb{R}^m, \quad \mu(q) = (-4\pi f_1(q), \dots, -4\pi f_m(q)) . \quad (2.12)$$

3. KÄHLER-RICCI SOLITONS AND ASSOCIATED VECTOR FIELDS ON HOMOGENEOUS TORIC BUNDLES

First of all, let us recall the definition of Kähler-Ricci soliton (see e.g. [16]).

Definition 3.1. Let $(N, J, \hat{\omega})$ be a compact Kähler manifold of positive first Chern class. We call *Kähler-Ricci soliton* any pair (ω, X) , where ω is a Kähler form on M and X is a (real) vector field on M such that:

$$a) \mathcal{L}_{JX}\omega = 0, \quad b) \rho - \omega = \mathcal{L}_X\omega = d(\iota_X\omega) .$$

If (ω, X) is a Kähler-Ricci soliton, we will say that ω is the *Kähler form of the soliton* and that X is the *associated vector field*.

From b) it is clear that a compact Kähler manifold admits a Kähler-Ricci soliton only if it is Fano.

We need now to introduce some notation regarding Kähler-Ricci solitons. For a given compact Kähler manifold (N, J) , we denote by $\text{Aut}(N, J)$ the group of all complex automorphisms of N , by $\text{Aut}(N, J)^\circ$ its connected component of the identity and by $R_u(N, J)$ its unipotent radical.

In the next statement, we collect some crucial facts on Kähler-Ricci solitons obtained by Tian and Zhu (see [16], Thm. A, [17], Prop. 3.1, Prop. 2.1, Thm. 3.2).

Theorem 3.2. *Let $(N, J, \hat{\omega})$ be a compact Kähler manifold with positive first Chern class and assume that it admits a Kähler-Ricci soliton (ω, X) . Denote also by $G^{(\omega)} \subset \text{Aut}(N, J)^\circ$ the subgroup of all isometries of ω . Then:*

- i) $G^{(\omega)}$ is a maximal compact subgroup of $\text{Aut}(N, J)^\circ$ and JX belongs to the center $\mathfrak{z}(\mathfrak{g}^{(\omega)})$;

ii) all Kähler-Ricci solitons (ω', X') on N are of the form

$$\omega' = \sigma^* \hat{\omega}, \quad X' = \sigma_*^{-1}(X)$$

for some $\sigma \in \text{Aut}(N, J)^o$.

Remark 3.3. From i) and ii) of the previous theorem, it follows immediately that N admits a Kähler-Ricci soliton (ω, X) if and only if, for a given maximal compact subgroup $\tilde{G} \subset \text{Aut}(N, J)^o$, there is a Kähler-Ricci soliton $(\omega^{(\tilde{G})}, X^{(\tilde{G})})$, where $\omega^{(\tilde{G})}$ is \tilde{G} -invariant and $JX^{(\tilde{G})} \in \mathfrak{z}(\tilde{\mathfrak{g}})$.

Let us now consider an homogeneous toric bundle $M = G^{\mathbb{C}} \times_{P, \tau} F$, with the fiber F acted on by the torus T^m (and hence by its complexification $(T^m)^{\mathbb{C}}$). The following lemma is crucial.

Lemma 3.4. *Let $M = G^{\mathbb{C}} \times_{P, \tau} F$ be a homogeneous toric bundle, $\tilde{G} \subset \text{Aut}(M)^o$ a maximal compact subgroup containing $G \times T^m$ and $\tilde{\mathfrak{g}} = \text{Lie}(\tilde{G})$. Then $\mathfrak{z}(\tilde{\mathfrak{g}}) \subset \mathfrak{t}$.*

Proof. By Blanchard's Lemma ([4]; see also [1], Prop. 1, p. 45), for any $Y \in \mathfrak{z}(\tilde{\mathfrak{g}})$ the group $A \stackrel{\text{def}}{=} \overline{\exp(\mathbb{R} \cdot Y)}$ is a compact, abelian subgroup of $Z(\tilde{G})$ consisting of fiber preserving biholomorphisms. This implies that A projects onto a compact, connected group A_V of biholomorphisms of V with $A_V \subseteq C_{\mathcal{A}}(G^{\mathbb{C}})$, where $\mathcal{A} \stackrel{\text{def}}{=} \text{Aut}(V, J_V)^o$. Now, if $\mathcal{A} = G^{\mathbb{C}}$, then A_V is trivial because $G^{\mathbb{C}}$ is semisimple. If $\mathcal{A} \supsetneq G^{\mathbb{C}}$, then the possible pairs $(\mathcal{A}, G^{\mathbb{C}})$ have been classified by Onishchik in [12] and it is easily checked that $C_{\mathcal{A}}(G^{\mathbb{C}})$ is trivial and hence $A_V = \{e\}$. This means that A fixes all fibers and that the restriction of A to $F = F|_{eK}$ commutes with the action of T^m .

On the other hand, by Demazure's Structure Theorem for toric manifold (see e.g. [11], p. 140), $\text{Aut}(F, J_F)^o$ is a linear algebraic group and T^m is a maximal algebraic torus of $\text{Aut}(F, J_F)^o$. This implies that for any $a \in A$ the biholomorphism $a|_F : F \rightarrow F$ coincides with some biholomorphism $t|_F$, $t \in T^m$ or, equivalently, that $a \circ t^{-1}|_F = \text{Id}$. Since both a and t commute with G , it follows that $a \circ t^{-1}|_{\pi^{-1}(gK)} = \text{Id}$ for any fiber $\pi^{-1}(gK) \in M$ and hence that $a = t$ and $\mathfrak{z}(\tilde{\mathfrak{g}}) \subset \mathfrak{t}$. ■

From Lemma 3.4 and Remark 3.3, we immediately obtain the following corollary.

Corollary 3.5. *An homogeneous bundle $M = G^{\mathbb{C}} \times_{P, \tau} F$ admits a Kähler-Ricci soliton if and only if there is a Kähler-Ricci soliton (ω, X) on M , where ω is a $G \times T^m$ -invariant and $X = J\hat{Y}$ for some $Y \in \mathfrak{t} = \text{Lie}(T^m)$.*

4. THE TIAN-ZHU INVARIANTS OF A HOMOGENEOUS TORIC BUNDLES

In [17], G. Tian and X.-H. Zhu proved that, on a given compact complex manifold (N, J) , a vector field X is the associated vector field of a Kähler-Ricci soliton (ω, X) only if a certain holomorphic invariant homomorphism

$$\mathcal{F}_X : \mathfrak{aut}(N, J) \rightarrow \mathbb{R}$$

vanishes identically. Such homomorphism \mathcal{F}_X is an analogue of the classical Futaki invariant $\mathcal{F} : \mathfrak{aut}(N, J) \rightarrow \mathbb{R}$ of (N, J) ([6]) and one has $\mathcal{F}_X = \mathcal{F}$ when $X = 0$. In the following, we will call such homomorphism the *Tian-Zhu invariant associated with X* .

In [17] the following important property has been proved.

Theorem 4.1. ([17], Prop. 2.1) *Let (N, J) be a compact complex Kähler manifold with $c_1(N) > 0$. For any maximal compact subgroup $\tilde{G} \subset \text{Aut}^o(N, J)$, there exists exactly one element $Y \in \mathfrak{z}(\tilde{\mathfrak{g}})$ (possibly equal to 0) so that $\mathcal{F}_{J\tilde{Y}}(\cdot)$ vanishes identically.*

Let us now consider the toric bundle M . By Lemma 3.4 and Theorem 4.1, if we consider a maximal compact subgroup $\tilde{G} \subset \text{Aut}(M, J)$ that contains $G \times T^m$, there exists exactly one $Y \in \mathfrak{t}$ so that the Tian-Zhu invariant $\mathcal{F}_X(\cdot)$ with $X = J\hat{Y}$ vanishes.

We need now to determine the explicit expression for \mathcal{F}_X when $X = J\hat{Y}$, for some $Y \in \mathfrak{t}$. Recall that, since (Z'_1, \dots, Z'_m) is a basis for \mathfrak{t} , any vector field of this kind is of the form

$$X^{(\lambda)} = \sum_{\ell=1}^m \lambda^\ell J \hat{Z}'_\ell \quad (4.1)$$

for some suitable $\lambda = (\lambda^1, \dots, \lambda^m) \in \mathbb{R}^m$.

Lemma 4.2. *Assume that M is Fano and let ω be a $G \times T^m$ -invariant Kähler form on M , with algebraic representative so that $Z_\omega|_F = \sum_i f_i Z_i + Z_V$ for some smooth functions $f_i : F \rightarrow \mathbb{R}$. For any vector field X on M such that $\mathcal{L}_{JX}\omega = 0$, then there exists a unique smooth real valued function $\theta^{(X)}$ such that*

$$\begin{cases} \mathcal{L}_X \omega = \frac{1}{4\pi} dd^c \theta^{(X)} = \frac{i}{2\pi} \partial \bar{\partial} \theta^{(X)} \\ \int_M e^{\theta^{(X)}} \omega^{n+m} = \int_M \omega^{n+m} . \end{cases} \quad (4.2)$$

If $X = X^{(\lambda)}$ is a vector field of the form (4.1), then the corresponding function $\theta^{(\lambda)}$ is $G \times T^m$ -invariant and the restriction of $\theta^{(\lambda)}|_F$ is

$$\theta^{(\lambda)}|_F = -4\pi \sum_j \lambda^j f_j + C^{(\lambda)} \quad (4.3)$$

where $C^{(\lambda)}$ is the real number

$$C^{(\lambda)} = \log \left(\frac{\int_M \omega^{n+m}}{\int_M e^{-4\pi \sum_i \lambda^i f_i} \omega^{n+m}} \right) . \quad (4.4)$$

The constant $C^{(\lambda)}$ is the same for all cohomologous $G \times T^m$ -invariant Kähler forms.

Proof. Since $c_1(M) > 0$ and $\mathcal{L}_{JX}\omega = d\iota_{JX}\omega = 0$, by Bochner's theorem $b_1(M) = 0$ and there exists a unique function θ_X so that

$$\frac{1}{4\pi} d^c \theta^{(X)} = (\iota_X \omega) \circ J = -\iota_{JX} \omega$$

and (4.2)₂ is satisfied. From uniqueness and the fact that ω and $X^{(\lambda)}$ are both $G \times T^m$ -invariant, it follows that the function $\theta^{(\lambda)}$ associated with $X = X^{(\lambda)}$ is

$G \times T^m$ -invariant. Moreover, one can check the $G \times T^m$ -invariant function defined by (4.3) is the required function because it satisfies (4.2)₂ and

$$dd^c \theta^{(\lambda)}(\hat{Z}_i, J\hat{Z}_j) \Big|_F = \mathcal{L}_{X^{(\lambda)}} \omega(\hat{Z}_i, J\hat{Z}_j) \Big|_F \quad 1 \leq i, j \leq m .$$

Finally, by (2.6) if ω' and ω are cohomologous, the algebraic representative of ω' is given by $Z_{\omega'}|_F = \sum_j f'_j Z_j + Z_V$ with $f'_j = f_j - J\hat{Z}_i(\frac{1}{4\pi}\phi)$ for some smooth $G \times T^m$ -invariant function $\phi : M \rightarrow \mathbb{R}$. By (4.3), the function $\theta'^{(\lambda)}$ relative to ω' is

$$\theta'^{(\lambda)} = -4\pi \sum_j \lambda^j f_j + \lambda^j J\hat{Z}_j(\phi) + C'^{(\lambda)} = \theta^{(\lambda)} + X^{(\lambda)}(\phi) + (C'^{(\lambda)} - C^{(\lambda)}) ,$$

where we denoted by $C'^{(\lambda)}$ the constant (4.4) determined by ω' in place of ω . On the other hand, Lemma 2.1 of [17] shows that $\theta'^{(\lambda)} = \theta^{(\lambda)} + X^{(\lambda)}(\phi)$ and hence that $C'^{(\lambda)} = C^{(\lambda)}$. ■

Proposition 4.3. *Assume that M is Fano. For any $\lambda \in \mathbb{R}^m$, we have that $\mathcal{F}_{X^{(\lambda)}}(J\hat{Y}) = 0$ for all $Y \in \mathfrak{t}$ if and only if all integrals*

$$\int_{\Delta_F} x_k e^{\lambda^a x_a} \prod_{\alpha \in R_m^+} \left(-\frac{a_\alpha^j x^j}{4\pi} + b_\alpha \right) dx^1 \wedge \cdots \wedge dx^m , \quad 1 \leq k \leq m , \quad (4.5)$$

vanish.

Proof. Let ω be a $G \times T^m$ -invariant Kähler form in $c_1(M)$, ρ the Ricci form of ω and h_ω a smooth function on M such that

$$\rho - \omega = \frac{1}{4\pi} dd^c h_\omega = \frac{i}{2\pi} \partial \bar{\partial} h_\omega . \quad (4.6)$$

It follows that the algebraic representatives of ρ , ω and $\frac{1}{4\pi} dd^c h_\omega$ are so that $Z_\rho - Z_\omega = Z_{\frac{1}{4\pi} dd^c h_\omega}$. From (2.6) - (2.8), we have that

$$J\hat{Z}_j(h_\omega) = -J\hat{Z}_j(\log h) + 4\pi f_j \quad (4.7)$$

where $h : F \rightarrow \mathbb{R}$ is defined in (2.9). According to the definition given in [17],

$$\mathcal{F}_{X^{(\lambda)}}(Y) \stackrel{\text{def}}{=} \int_M Y(h_\omega - \theta^{(\lambda)}) e^{\theta^{(\lambda)}} \omega^{n+m} \quad \text{for any } Y \in \mathfrak{aut}(M, J) , \quad (4.8)$$

where $\theta^{(\lambda)} : M \rightarrow \mathbb{C}$ is the unique smooth function that satisfies the condition (4.2).

Let us now compute $\mathcal{F}_{X^{(\lambda)}}(J\hat{Y})$ when $Y = Z'_k$. First of all, let us fix a point $p_o \in F_{\text{reg}}$ and consider the diffeomorphism

$$\xi : F_{\text{reg}} \rightarrow \mathbb{R}^m \times T^m \simeq (\mathbb{C}_*)^m ,$$

$$\xi \left(\exp \left(\sum_{j=1}^m (t^j + is^j) iZ'_j \right) \cdot p_o \right) = \left(\frac{2\pi}{\nu_1} t^1 e^{i \frac{2\pi}{\nu_1} s^1}, \dots, \frac{2\pi}{\nu_m} t^m e^{i \frac{2\pi}{\nu_m} s^m} \right) .$$

If we identify F_{reg} with $(\mathbb{C}_*)^m$ by means of ξ , the pairs $(\frac{2\pi}{\nu_i} t^i, \frac{2\pi}{\nu_i} s^i)$'s are polar coordinates for the factors \mathbb{C}_* of $(\mathbb{C}_*)^m$ and we may consider the m -tuple $(\frac{2\pi}{\nu_1}(t^1 + is^1), \dots, \frac{2\pi}{\nu_m}(t^m + is^m))$ as a system of complex coordinates on $F_{\text{reg}} \simeq (\mathbb{C}_*)^m$ such that

$$\frac{\partial}{\partial t^i} = J\hat{Z}_i , \quad \frac{\partial}{\partial s^i} = -\hat{Z}_i .$$

Now, set $\Omega_F = dt^1 \wedge \cdots \wedge dt^m \wedge ds^1 \wedge \cdots \wedge ds^m$.

Lemma 4.4. *There is a suitable constant C such that for any G -invariant function $\phi \in \mathcal{C}^\infty(M)^G$ we have*

$$\int_M \phi \cdot \omega^{n+m} = C \cdot \text{Vol}_{\omega_V^n}(V) \cdot \int_{F_{\text{reg}}} \phi \cdot h \cdot \Omega_F, \quad (4.9)$$

where $h \in \mathcal{C}^\infty(F)^{T^m}$ is the T^m -invariant function defined in (2.9).

Proof. Let $\mathcal{U}_{\mathfrak{m}}$ be an open set in \mathfrak{m} containing 0 such that the map $\psi : \mathcal{U}_{\mathfrak{m}} \rightarrow \exp(\mathcal{U}_{\mathfrak{m}}) \cdot (eP) \stackrel{\text{def}}{=} \mathcal{U}_V$ is a diffeomorphism onto its image and the mapping $\lambda : F \times \mathcal{U}_V \rightarrow \pi^{-1}(\mathcal{U}_V) \stackrel{\text{def}}{=} \mathcal{U}$ given by $\lambda(f, \exp(X) \cdot (eP)) = \exp(X) \cdot f$ is a bundle isomorphism. We then select $g_j \in G$, $j = 1, \dots, N$, so that $V = \bigcup_{j=1}^N (g_j \cdot \mathcal{U}_V)$ and put $A_0 = \emptyset$ and $A_j \stackrel{\text{def}}{=} g_j \mathcal{U}$ for $j = 1, \dots, N$. Hence

$$\begin{aligned} \int_M \phi \cdot \omega^{n+m} &= \sum_{j=1}^N \int_{A_j \setminus (\bigcup_{i=0}^{j-1} A_i)} \phi \cdot \omega^{n+m} = \\ &= \sum_{j=1}^N \int_{\mathcal{U} \setminus (\bigcup_{i=0}^{j-1} g_j^{-1} A_i)} \phi \cdot \omega^{n+m}. \end{aligned} \quad (4.10)$$

In $\mathcal{U} \cong F \times \mathcal{U}_V$ we may restrict to the submanifold $F_{\text{reg}} \times \mathcal{U}_V$ and we may define the function $\tilde{h} \in C^\infty(F_{\text{reg}} \times \mathcal{U}_V)$ by means of the following

$$\omega^{n+m} = \tilde{h} \cdot \Omega_F \wedge \omega_V^n.$$

The function \tilde{h} can be easily determined by evaluating the forms ω^{n+m} and $\Omega_F \wedge \omega_V^n$ on the frames $\{\hat{Z}_i, J\hat{Z}_i, \hat{F}_j, \hat{G}_j\}$ at the points of F_{reg} ; A direct computation shows that $\tilde{h} = C \cdot h$, for some suitable constant C .

Using the G -invariance and Fubini's theorem, (4.10) reads

$$\begin{aligned} \int_M \phi \cdot \omega^{n+m} &= \sum_{j=1}^N \int_{F \times (\mathcal{U}_V \setminus (\bigcup_{i=1}^{j-1} g_i^{-1} g_j \mathcal{U}_V))} \phi \cdot \omega^{n+m} = \\ &= \sum_{j=1}^N \int_{(\mathcal{U}_V \setminus (\bigcup_{i=1}^{j-1} g_i^{-1} g_j \mathcal{U}_V))} \left(\int_{F_{\text{reg}}} \phi \cdot \tilde{h} \cdot \Omega_F \wedge \omega_V^n \right) = \\ &= C \left(\sum_{j=1}^N \int_{\mathcal{U}_V \setminus (\bigcup_{i=1}^{j-1} g_i^{-1} g_j \mathcal{U}_V)} \omega_V^n \right) \cdot \int_{F_{\text{reg}}} \phi \cdot h \cdot \Omega_F = C \cdot \text{Vol}_{\omega_V^n}(V) \cdot \int_{F_{\text{reg}}} \phi \cdot h \cdot \Omega_F. \quad \blacksquare \end{aligned}$$

By Lemma 4.4, (4.8), (4.2) and (4.7), it follows that $\mathcal{F}_{X^{(\lambda)}}(J\hat{Y}) = 0$ for all $Y \in \mathfrak{t}$ if and only if the integrals

$$\begin{aligned} g_k^{(\lambda)} &= \int_{F_{\text{reg}}} J\hat{Z}_k \left(h_\omega - \theta^{(\lambda)} \right) e^{\theta^{(\lambda)}} \cdot h \cdot \Omega_F = \\ &= \int_{F_{\text{reg}}} \left(-J\hat{Z}_k(\log h) + 4\pi f_k + J\hat{Z}_k(4\pi \lambda^j f_j) \right) \cdot e^{-4\pi \lambda^j f_j + C^{(\lambda)}} \cdot h \cdot \Omega_F \end{aligned} \quad (4.11)$$

are equal to 0 for all $k = 1, \dots, m$.

On the other hand, if we identify F_{reg} with $(\mathbb{C}_*)^m$ by means of the map ξ described above, we have that

$$\begin{aligned}
& \int_{F_{\text{reg}}} \left(J\hat{Z}_k(\log h) - J\hat{Z}_k(4\pi\lambda^j f_j) \right) \cdot e^{-4\pi\lambda^j f_j} \cdot h \cdot \Omega_F = \\
& = \int_{\mathbb{R}^m} \frac{\partial}{\partial t^k} \left(h \cdot e^{-4\pi\lambda^j f_j} \right) dt^1 \wedge \dots \wedge dt^m \wedge ds^1 \wedge \dots \wedge ds^m = \\
& = (-1)^k \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial}{\partial t^k} \left(h \cdot e^{-4\pi\lambda^j f_j} \right) dt^k \right) dt^1 \wedge \dots \wedge_k \dots \wedge dt^m \wedge ds^1 \wedge \dots \wedge ds^m = \\
& = (-1)^k \int_{\mathbb{R}^{m-1}} \left(\lim_{\substack{a \rightarrow +\infty \\ b \rightarrow -\infty}} \left(h \cdot e^{-4\pi\lambda^j f_j} \right) \Big|_{t^k=a}^{t^k=b} \right) dt^1 \wedge \dots \wedge_k \dots \wedge dt^m \wedge ds^1 \wedge \dots \wedge ds^m = \\
& = 0
\end{aligned} \tag{4.12}$$

where the last equality is obtained from the definition of h , the fact that the functions $f_j : F_{\text{reg}} \rightarrow \mathbb{R}$ are bounded (see (2.12)) and the property that

$$\det(-f_{i,j}) : F_{\text{reg}} \rightarrow \mathbb{R}, \quad \det(-f_{i,j})|_p = \frac{1}{(2\pi)^m} \det(g_p(\hat{Z}_i, \hat{Z}_j))$$

goes to 0 when p tends to a point of $F \setminus F_{\text{reg}}$, since

$$F \setminus F_{\text{reg}} = \{ q \in F \text{ such that } \hat{Z}_j|_q = 0 \text{ for some } j = 1, \dots, m \}.$$

From (4.12), it follows that the integrals $g_k^{(\lambda)}$ are equal to

$$g_k^{(\lambda)} = C \int_{F_{\text{reg}}} f_k e^{-4\pi\lambda^j f_j} \det(-f_{\ell,s}) \cdot \prod_{\alpha \in R_m^+} (a_\alpha^i f_j + b_\alpha) \cdot \Omega_F \tag{4.13}$$

for some constant C . Using the change of variables $(t^i, s^i) \mapsto (x^i = -4\pi f^i(t^j, s^k), s^i)$ and the fact that the integrand is independent of the coordinates s^i , one can check that (4.13) is equal (up to a multiplicative constant) to the integral (4.5) over the image of the moment map $\mu = (-4\pi f_1, \dots, -4\pi f_m)$, i.e. the canonical polytope $\Delta_F \subset \mathbb{R}^m$. ■

5. THE REDUCTION OF THE SOLITONIC KÄHLER EQUATION ON M TO AN EQUATION ON THE TORIC MANIFOLD F AND THE PROOF OF THEOREM 1.1

By Thm. 1.1 of [14], we know that $M = G^{\mathbb{C}} \times_{P,\tau} F$ is Fano only if also F is Fano. In the proof of that theorem, we have shown that the correspondence $\omega \mapsto \omega|_{TF}$ between 2-forms on M and on F maps any G -invariant Kähler form in $c_1(M)$ into a T^m -invariant Kähler form on F , which belongs to $c_1(F)$. The following lemma shows that such correspondence is actually bijective.

Lemma 5.1. *Let $c_1(M) > 0$ and denote by $c_1(M)^{G \times T^m}$ and $c_1(F)^{T^m}$ the sets of $G \times T^m$ -invariant 2-forms in $c_1(M)$ and of T^m -invariant 2-forms in $c_1(F)$, respectively. Then there exists a map*

$$E : c_1(F)^{T^m} \longrightarrow c_1(M)^{G \times T^m} \quad (5.1)$$

which is inverse to the map

$$R : \omega \in c_1(M)^{G \times T^m} \longrightarrow \omega|_{TF} \in c_1(F)^{T^m} .$$

Moreover, $E(\omega)$ is Kähler if and only if ω is Kähler.

Proof. Let us fix a Kähler form $\omega_o \in c_1(F)^{T^m}$. Any T^m -invariant $\omega \in c_1(F)$ is of the form $\omega = \omega_o + dd^c \phi_\omega$ for some T^m -invariant function ϕ_ω , which is unique up to a constant. We denote by μ_ω the map

$$\mu_\omega : F \rightarrow \mathfrak{t}^* , \quad \mu_\omega|_p(X) \stackrel{\text{def}}{=} \mu_{\omega_o}|_p(X) - d^c \phi_\omega(\hat{X}_p) \text{ for any } X \in \mathfrak{t} . \quad (5.2)$$

where μ_{ω_o} is the metrically normalized moment map associated to the Kähler form ω_o (for the definition, see §2). One can check that μ_ω is the metrically normalized moment map relative to ω , whenever ω is non-degenerate.

Now, for any $\omega \in c_1(F)^{T^m}$ we define $E(\omega)$ as the unique $G \times T^m$ -invariant 2-form on M , whose restriction on $TM|_F$ is as follows: for any $p \in F$, $X, Y \in T_p F$ and $A, B \in \mathfrak{m}$

$$\begin{aligned} E(\omega)_p(X, Y) &= \omega_p(X, Y) , & E(\omega)_p(X, \hat{A}) &= 0 , \\ E(\omega)_p(\hat{A}, \hat{B}) &= -\mu_\omega(p)(\tau([A, B]_{\mathfrak{k}})) + (\pi^* \omega_V)_p(\hat{A}, \hat{B}) , \end{aligned} \quad (5.3)$$

where ω_V is the G -invariant Kähler-Einstein form on V with Einstein constant $c = 1$ and where we denoted by “ $[A, B]_{\mathfrak{k}}$ ” the component of $[A, B]$ along \mathfrak{k} w.r.t. the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Going through the arguments after formula (5.1) of [14], one can check that $E(\omega)$ is closed and J -invariant.

It is also direct to see that the algebraic representative $Z_{E(\omega)}$ of $E(\omega)$ is so that

$$Z_{E(\omega)}|_F = \sum_j \left(f_{oj} - J \hat{Z}_j(\phi_\omega) \right) Z_j + Z_V \quad (5.4)$$

where the $f_{oj} : F \rightarrow \mathbb{R}$ are (up to the factor -4π) equal to the components of $\mu_{\omega_o} : F \rightarrow \mathfrak{t}^* \simeq \mathbb{R}^m$ under the identification (2.12). It follows that the algebraic representative of $E(\omega_2) - E(\omega_1)$ is the same of $dd^c(\phi_{\omega_2} - \phi_{\omega_1})$, meaning that the image of E is in a single cohomology class. Moreover, by looking at the algebraic representatives, one can see that the 2-form $E(\rho)$, where ρ is the Ricci form of ω , coincides with the 2-form ρ_o defined in formula (5.7) of [14]. By the proof of Thm. 1.1 in [14], we know that $\rho_o \in c_1(M)$ and hence $E(\omega) \in c_1(M)$ for any ω .

From (5.4) and the remarks at the end of §2, for any $\omega \in c_1(M)^{G \times T^m}$ the algebraic representatives of ω and of $E(R(\omega))$ coincide and hence $\omega = E(R(\omega))$. This implies that E is inverse to R since, by construction, we also have that $R(E(\omega)) = \omega$ for any $\omega \in c_1(F)^{T^m}$.

The last claim follows from the fact that, for any Kähler form $\omega \in c_1(F)^{T^m}$, the 2-form $E(\omega)$ is positive because it is G -invariant and its restriction at $TM|_F$ is positive. This is true because, if we denote by $-4\pi f_j$ the components of the

metrically normalized moment map μ_ω of ω , from (5.4) we have that for any $\alpha_j \in R_m^+$

$$\tilde{\omega}_p(\hat{F}_{\alpha_j}, J\hat{F}_{\alpha_j}) = \mathcal{B}(\sum_k f_k Z_k + Z_V, [F_{\alpha_j}, G_{\alpha_i}]) = i\alpha_j(\sum_k f_k Z_k + Z_V) > 0, \quad (5.5)$$

which coincides with condition (1.1) of Thm. 1.1 of [14] in a different notation. ■

We now want to determine the differential equations that characterize the T^m -invariant Kähler forms in $c_1(F)$ corresponding to invariant solitonic Kähler forms of M . We recall that, by Corollary 3.5, there exists a solitonic Kähler form on M if and only if there exists a Kähler-Ricci soliton $(\tilde{\omega}, X)$ where $\tilde{\omega}$ is $G \times T^m$ -invariant and X is of the form $X = X^{(\lambda)} = \sum_k \lambda^k J\hat{Z}'_k$ for some $\lambda \in \mathbb{R}^m$.

To simplify the notation, in the following, for any T^m -invariant 2-form $\omega \in c_1(F)$, we will denote by $\tilde{\omega} = E(\omega)$ the corresponding 2-form in $c_1(M)^{G \times T^m}$. Similarly, any 2-form in $c_1(M)^{G \times T^m}$ will be denoted with a symbol of the form $\tilde{\omega}$ and the corresponding 2-form in $c_1(F)$ will be indicated by $\omega = R(\tilde{\omega})$.

For any Kähler form $\tilde{\omega} \in c_1(M)^{G \times T^m}$, let us indicate by $(-4\pi) \cdot f_{\omega,j} : F \rightarrow \mathbb{R}$ the components of the metrically normalized moment map $\mu_\omega : F \rightarrow \Delta_F \subset \mathbb{R}^m \simeq \mathfrak{t}^*$, relative to ω , as given in (2.11) and (2.12). Let also $\tilde{\rho}$ be the Ricci form of $\tilde{\omega}$ and $\phi_{\tilde{\omega}, \tilde{\rho}}$ the unique potential on M so that

$$\tilde{\rho} = \tilde{\omega} + \frac{1}{4\pi} dd^c \phi_{\tilde{\omega}, \tilde{\rho}}, \quad \int_M e^{\phi_{\tilde{\omega}, \tilde{\rho}}} \tilde{\omega}^{n+m} = \int_M \tilde{\omega}^{n+m}. \quad (5.6)$$

Notice that $\phi_{\tilde{\omega}, \tilde{\rho}}$ is $G \times T^m$ -invariant and hence it is uniquely determined by its restriction to F .

From Lemma 4.2, the pair $(\tilde{\omega}, X^{(\lambda)})$ is a Kähler-Ricci soliton if and only if

$$\phi_{\tilde{\omega}, \tilde{\rho}} + 4\pi \sum_{\lambda}^i j f_{\omega,j} - C^{(\lambda)} = 0, \quad (5.7)$$

where $C^{(\lambda)}$ is defined by (4.4) and it is independent of ω .

We now fix a T^m -invariant Kähler form $\omega_o \in c_1(F)$. For any other T^m -invariant Kähler form ω , we denote by ψ_ω the unique potential on F so that

$$\omega = \omega_o + \frac{1}{4\pi} dd^c \psi_\omega, \quad \int_F e^{\psi_\omega} \omega_o^m = \int_F \omega_o^m \quad (5.8)$$

and we want to determine the equation in the unknown function ψ_ω determined by the condition (5.7).

Let $\tilde{\omega}_o = E(\omega_o)$ be the $G \times T^m$ -invariant Kähler form given by Lemma 5.1 and let $Z_{\tilde{\omega}_o} = \sum_i f_{oi} Z_i + Z_V$ be the restriction to F of the algebraic representative of $\tilde{\omega}_o$. We consider also a system of complex coordinates $(t^1 + is^1, \dots, t^m + is^m)$ on $F_{\text{reg}} \simeq (\mathbb{C}_*)^m$ as in the proof of Proposition 4.3, such that

$$\frac{\partial}{\partial t^i} = J\hat{Z}_i, \quad \frac{\partial}{\partial s^i} = -\hat{Z}_i. \quad (5.9)$$

Since $F_{\text{reg}} \simeq \mathbb{R}^m \times T^m$, the maps f_{oi} are T^m -invariant and, by (2.10), $\frac{\partial f_{oi}}{\partial t^j} = \frac{\partial f_{oj}}{\partial t^i}$ for all i, j 's, then there exists a T^m -invariant smooth function $u_o : F_{\text{reg}} \rightarrow \mathbb{R}$ so that

$$f_{oi}|_{F_{\text{reg}}} = -\frac{1}{4\pi} \frac{\partial u_o}{\partial t^i}. \quad (5.10)$$

The function u_o is uniquely determined up to an additive constant. It can also be checked that $\omega_o|_{F_{\text{reg}}} = \frac{1}{4\pi}dd^c u_o$.

We claim that there exists some suitable constant C_ω so that $\phi_{\tilde{\omega}, \tilde{\rho}}|_{F_{\text{reg}}} = \Psi + C_\omega$ where

$$\Psi \stackrel{\text{def}}{=} -\log \left| \det \left(\frac{\partial^2(u_o + \psi_\omega)}{\partial t^i \partial t^j} \right) \cdot \prod_{\alpha \in R_m^+} \left(-\frac{a_\alpha^i}{4\pi} \frac{\partial(u_o + \psi_\omega)}{\partial t^i} + b_\alpha \right) \right| - (u_o + \psi_\omega) . \quad (5.11)$$

In order to check this, notice that by (2.6), the restriction to F of the algebraic representative of $\tilde{\omega} = \tilde{\omega}_o + \frac{1}{4\pi}dd^c \psi_\omega$ is

$$Z_{\tilde{\omega}} = \sum_j \left(f_{oj} - \frac{1}{4\pi} J Z_j(\psi_\omega) \right) Z_j + Z_V = -\frac{1}{4\pi} \sum_j \frac{\partial(u_o + \psi_\omega)}{\partial t^j} Z_j + Z_V .$$

Then, from (2.8), it follows that the algebraic representative of $\frac{1}{4\pi}dd^c \Psi$ coincides with the algebraic representative of $\tilde{\rho} - \tilde{\omega} = \frac{1}{4\pi}dd^c \phi_{\tilde{\omega}, \tilde{\rho}}$ and hence that $\Psi - \phi_{\tilde{\omega}, \tilde{\rho}}$ is a constant. The value of C_ω is uniquely determined by the normalizing condition (5.6)₂.

From (5.7) together with the expression of $\phi_{\tilde{\omega}, \tilde{\rho}}$ given by (5.11) and setting $\varphi = \psi_\omega + C_\omega$, we obtain the following proposition, which reduces the solitonic equation on M to a Monge-Ampere equation on the toric manifold F .

Proposition 5.2. *Let ω_o be a fixed T^m -invariant Kähler form in $c_1(F)$ and $u_o : F_{\text{reg}} \rightarrow \mathbb{R}$ be a fixed smooth function so that (5.10) holds. Let also $X^{(\lambda)}$ be the vector field of the form (4.1) such that $\mathcal{F}_{X^{(\lambda)}}(\cdot) = 0$ and let $C^{(\lambda)}$ be the constant defined by (4.4).*

A T^m -invariant Kähler metric $\omega \in c_1(F)$ is so that the corresponding $G \times T^m$ -invariant Kähler form $\tilde{\omega} \in c_1(M)$ is a soliton, with associated vector field $X^{(\lambda)}$, if and only if

$$\omega = \omega_o + \frac{1}{4\pi}dd^c \varphi ,$$

where $\varphi : F \rightarrow \mathbb{R}$ is a smooth T^m -invariant function that satisfies

$$\det \left(\frac{\partial^2(u_o + \varphi)}{\partial t^i \partial t^j} \right) = \frac{1}{\prod_{\alpha \in R_m^+} \left(-\frac{a_\alpha^i}{4\pi} \frac{\partial(u_o + \varphi)}{\partial t^i} + b_\alpha \right)} e^{-C^{(\lambda)} - X^{(\lambda)}(u_o + \varphi) - u_o - \varphi} \quad (5.12)$$

at all points of F_{reg} .

We recall that, by Theorem 4.1 and Lemma 3.4, there exists a unique $\lambda \in \mathbb{R}^m$ so that $\mathcal{F}_{X^{(\lambda)}}(\cdot)$ vanishes identically. By Proposition 4.3, for such λ all integrals (4.5) are equal to 0.

In [19], X.-J. Wang and X. Zhu determined a Monge-Ampere equation that characterizes the T^m -invariant Kähler-Ricci solitons on F and proved the solvability of such equation. Notice that the case considered by Wang and Zhu can be interpreted as a homogeneous toric bundle with basis given by a single point. And in fact the Monge-Ampere of Wang and Zhu can be obtained from (5.12) by setting the factor

$$\mathcal{A} = \frac{1}{\prod_{\alpha \in R_m^+} \left(-\frac{a_\alpha^i}{4\pi} \frac{\partial(u_o + \varphi)}{\partial t^i} + b_\alpha \right)} \quad (5.13)$$

equal to 1.

We claim that the arguments used in [19] for proving the solvability of (5.12) when $\mathcal{A} = 1$ remain valid also when $\mathcal{A} \neq 1$ and hence that (5.12) is always solvable.

In fact, as in [19], the solvability of (5.12) can be obtained by the continuity method, namely by considering the family of equations

$$\det \left(\frac{\partial^2 (u_o + \varphi)}{\partial t^i \partial t^j} \right) = \frac{1}{\prod_{\alpha \in R_m^+} \left(-\frac{a_\alpha^i}{4\pi} \frac{\partial (u_o + \varphi)}{\partial t^i} + b_\alpha \right)} e^{-C^{(\lambda)} - X^{(\lambda)}(u_o + \varphi) - u_o - t\varphi}, \quad (5.14)$$

parameterized by the real numbers $t \in [0, 1]$. By the same arguments used for the proof of Proposition 5.2, one can see that (5.14) is the Monge-Ampere equation characterizing the T^m -invariant 2-forms $\omega \in c_1(F)$, whose associated $G \times T^m$ -invariant 2-form $\tilde{\omega}$ satisfy

$$\tilde{\rho} - \tilde{\omega}_o - t(\tilde{\omega} - \tilde{\omega}_o) = \mathcal{L}_{X^{(\lambda)}} \tilde{\omega}. \quad (5.15)$$

By the results of [20] and [16], (5.15) is solvable for any t in an open subinterval $[0, \epsilon[\subset [0, 1]$ and hence the same is true for the equation (5.14). Moreover, for any $t \in [0, 1]$, if φ is a solution of (5.14) that corresponds to a Kähler metric, then its $G \times T^m$ -invariant extension is the potential w.r.t. $\tilde{\omega}_o$ of a Kähler form $\tilde{\omega}$ that satisfies (5.15) and hence in $c_1(M)$. It follows that the functions

$$-4\pi f_i = \frac{\partial (u_o + \varphi)}{\partial t^i}$$

are the components of the metrically normalized moment map of ω and take value in the polytope Δ_F , which is a bounded convex domain in \mathbb{R}^m and it is independent of t . In particular, the algebraic representative of $\tilde{\omega}$ is of the form (2.7), the function $\mu = (-4\pi f_1, \dots, -4\pi f_m) : F \rightarrow \mathbb{R}^m \simeq t^*$ is a metrically normalized moment map on F and the image $\mu(F) = \Delta_F$ is independent of the value of t . Since \mathcal{A} coincides with the function $\prod_{\alpha \in R_m^+} (-\mathcal{B}(\sum_{i=1}^m f_i Z_i + Z_V, H_\alpha))$, by Thm 1.1 of [14], the values of \mathcal{A} are always positive and are bounded above and below by two constants that are independent of t .

From this facts and the generalizations of [16] of the a priori estimates of [18], the solvability of (5.14) for $t = 1$ is proved if one can give a uniform upper and lower estimates for the solutions φ of (5.14) for any $t \in [\epsilon, 1]$ for some $0 < \epsilon$.

One can check that the proofs of Lemmata 3.2 - 3.5 in [19] remain valid also if $\mathcal{A} \neq 1$ provided that the following properties and differences on notation are taken into account:

- i) The canonical polytope Δ_F should be considered as equal to the dual Ω^* of the polytope denoted by Ω in [19] and the m -tuple $(\lambda^1, \dots, \lambda^m)$ should be considered as equal to the m -tuple of constants (c_1, \dots, c_m) considered in [19]; Moreover, with no loss of generality, one should assume that our function $u_o : F_{\text{reg}} \rightarrow \mathbb{R}$ so that $\omega_o|_{TF} = \frac{1}{4\pi} dd^c u_o$ coincides with the function denoted by “ u^0 ” in [19];
- ii) By the previous remarks, if M is Fano, there exist two positive real numbers $0 < K_1 < K_2$, independent on t , so that $K_1 \leq \mathcal{A} \leq K_2$ at any point of F_{reg} and for any solution of (5.12);
- iii) From (ii) and the fact that all integrals (4.5) are equal to 0, the polytope $\Omega^* = \Delta_F \subset \mathbb{R}^m$ contains the origin also when $\mathcal{A} \neq 1$;

- iv) The equality $0 = \int_{\Omega^*} y_i e^{\sum_{\ell} c_{\ell} y_{\ell}} dy$ which appears at the end of the proof of Lemma 3.3 of [19] should be replaced by the equality

$$0 = \int_{\Omega^*} y_i e^{\sum_{\ell} c_{\ell} y_{\ell}} \prod_{\alpha \in R_m^+} \left(-\frac{\sum_j a_{\alpha}^j y^j}{4\pi} + b_{\alpha} \right) dy^1 \wedge \cdots \wedge dy^m$$

which is true by Proposition 4.3; Under this replacement, all remaining equalities considered in Lemma 3.3 of [19] remain true also when $\mathcal{A} \neq 1$.

Those lemmata give the needed estimates and the solvability of (5.12) is proved.

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